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Conforming Chebyshev Spectral Collocation Methods
for the Solution of Laminar Flow
in a Constricted Channel

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Abstract

The numerical simulation of steady planar two-dimensional, laminar flow of an incompressible fluid through an abruptly contracting channel using spectral domain decomposition methods is described. The key features of the method are the decomposition of the flow region into a number of rectangular subregions and spectral approximations which are pointwise C^1 continuous across subregion interfaces. Spectral approximations to the solution are obtained for Reynolds numbers in the range [0, 500]. The size of the salient corner vortex decreases as the Reynolds number increases from 0 to around 45. As the Reynolds number is increased further the vortex grows slowly. A vortex is detected downstream of the contraction at a Reynolds number of around 175 that continues to grow as the Reynolds number is increased further.

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1. Introduction

The numerical simulation of the flow in a constricted channel is considered using spectral domain decomposition techniques. The first numerical solutions to this problem were obtained by Dennis and Smith (1980) using a finite difference discretization of the stream function - vorticity formulation of the governing equations. They did not detect a downstream recirculation region caused by the flow separating at the corner despite using a very fine uniform grid. In a recent paper Hunt (1989) uses a non-uniform grid to ensure a dense distribution of grid points in the regions of the flow which need to be resolved. He places a locally fine mesh downstream of the constriction and around the re-entrant corner since this gives rise to a singularity in the vorticity.

In this paper the governing equations are written in terms of the stream function. This means that mass is conserved identically. The flow region is divided into a number of rectangular subdomains. In each of these subdomains the stream function is approximated by a truncated double Chebyshev expansion. The expansion coefficients are determined by collocating the governing equation and appropriate interface continuity conditions between subdomains. The decomposition is such that the Chebyshev collocation points are distributed densely around the corner and near solid boundaries. This is important computationally in order to resolve the main features of the flow efficiently.

In previous work, Karageorghis and Phillips (1989a) consider nonconforming subdomains because of their ease of implementation. Although this strategy works well for the Stokes problem, a lack of interface continuity appears for the Navier-Stokes problem (Karageorghis and Phillips (1989b)) for values of the Reynolds number around 200, eventually causing the method to break down completely. The spectral approximations obtained are not pointwise continuous across the subdomain interface. In this paper a collocation strategy is used which generates pointwise C^1 continuous approximations across the interfaces. As a result no computational limit on the value of the Reynolds number has yet been encountered. Numerical solutions are obtained for values of the Reynolds number up to 500.

The matrices arising from spectral discretizations are not sparse like their

finite difference and finite element counterparts. However, when used in conjunction with domain decomposition techniques these matrices possess a block tridiagonal structure which can be exploited when designing methods of solution. Here we use a subroutine from the NAG Library which solves almost block diagonal systems (Brankin and Gladwell (1990)). This method is as efficient as the capacitance matrix method in terms of cost and storage but is found to be more stable (Karageorghis and Phillips (1990)).

Spectral approximations to the solution of this problem are obtained for Reynolds numbers in the range [0, 500]. A comparison with the work of Dennis and Smith (1980) shows good agreement between the sets of results in terms of stream function contours in the bulk of the flow and the description of the salient corner vortex. Although they did not find a downstream recirculation region there is a hint of its existence on their finest glides for Reynolds numbers above 1000. They probably failed to detect this behaviour in their numerical simulations because the grid employed was not fine enough in this region. In our calculations separation first appears at around Re = 175 which is slightly earlier than Hunt (1989) predicts. The separation length and the strength of the vortex increase as Re increases, the length being roughly proportional to Re. The spectral collocation method predicts longer separation lengths than Hunt (1989). On the finest grids used the spectral collocation method employs about a third of the number of degrees of freedom as Hunt (1989) and a fifteenth of the number of degrees of freedom as Dennis and Smith (1980).

2. The Governing Equations

We consider the steady laminar flow of an incompressible fluid through an abruptly contracting channel with walls at $y = \pm 1$ for $x < 0$, $y = \pm \frac{1}{2}$ for $x > 0$ and $1/2 \leq |y| \leq 1$ for $x = 0$. Upstream we impose parabolic Poiseuille flow and we suppose that the flow is parabolic again far enough downstream. Since the flow is symmetric about $y = 0$ it is only necessary to seek a solution for $y \geq 0$.

In terms of non-dimensional variables the incompressible Navier-Stokes equations are

$$(\underline{v} \cdot \nabla)\underline{v} = -\nabla p + (\mathrm{Re})^{-1} \nabla^2 \underline{v} \ , \tag{2.1}$$

$$\nabla \cdot \underline{v} = 0 \, , \tag{2.2}$$

where $\underline{v} = (u, v)$ is the velocity vector, p is the pressure and Re is the Reynolds number. The introduction of a stream function, $\psi(x, y)$, defined by

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x} \, , \tag{2.3}$$

means that the continuity equation (2.2) is satisfied identically and (2.1) becomes, after elimination of the pressure,

$$\nabla^4 \psi - \mathrm{Re}\left[\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \psi)\right] = 0 \tag{2.4}$$

The boundary conditions are

$$\psi = 1, \ \frac{\partial \psi}{\partial y} = 0 \ \text{ on } y = 1, \ x \leq 0 \ \text{ and } y = \frac{1}{2}, \ x \geq 0, \tag{2.5}$$

$$\psi = 1, \ \frac{\partial \psi}{\partial y} = 0 \ \text{ on } x = 0, \ \ \frac{1}{2} \leq y \leq 1 \, , \tag{2.6}$$

$$\psi = 0, \ \frac{\partial^2 \psi}{\partial y^2} = 0 \ \text{ on } y = 0. \tag{2.7}$$

The conditions on $y = 0$ preserve the symmetry of the flow while the other conditions represent the no-slip conditions.

The governing equation (2.4) is a nonlinear partial differential equation for the stream function. It is solved in an iterative fashion using a Newton-type method to linearize it (Phillips (1984)). It is the linearized partial differential equation which is solved at each Newton step using a spectral collocation method. Let us write (2.4) in operator form,

$$L(\psi) = 0 \tag{2.8}$$

where L is a nonlinear operator. Suppose that ψ^* is some approximation to the solution of (2.8). To obtain an improved approximation we replace L by its linearization about ψ^* and then solve the problem

$$L'(\psi^*)\phi = -L(\psi^*) \, , \tag{2.9}$$

where $L'(\psi)$ is the Frèchet derivative of L at ψ defined by

$$L'(\psi)\phi = \nabla^4\phi - Re\left[\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\phi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\phi) + \frac{\partial}{\partial x}(\nabla^2\psi)\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}(\nabla^2\psi)\frac{\partial\phi}{\partial x}\right] \tag{2.10}$$

The new approximation to the solution is thus $\psi^* + \phi$. This completes a single Newton step and is repeated until convergence is reached.

3. Domain Decomposition and Spectral Approximation

Since spectral methods are most easily applied to problems defined in rectangular regions, the natural domain decomposition of the flow region comprises three rectangular subdomains with common point (0, 1/2) as shown in Figure 1. Such a domain decomposition may be termed conforming since the sides of each rectangle are contiguous with either a boundary segment or a side of another rectangle. A further advantage of this decomposition is that it is possible to construct a collocation scheme which results in pointwise C^1 continuous approximations across the subdomain interfaces.

The flow region is truncated upstream and downstream of the constriction at finite distances h_1 and h_2 from the origin, respectively. The distances h_1 and h_2 need to be sufficiently large so that the flow is fully developed in the entry and exit sections. The domain truncation means that additional boundary conditions need to be imposed on entry and exit, namely that the normal derivative of ψ vanishes on entry and exit.

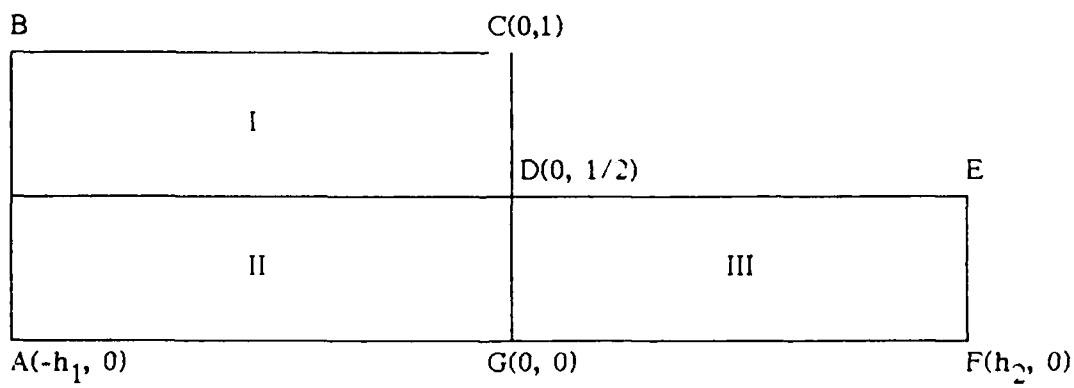

Fig. 1 Three subdomain decomposition of flow region.

In each subdomain the stream function $\psi(x, y)$ is approximated by $\psi^k(x, y)$, where

$$\psi^k(x, y) = G^k(y) + \sum_{m=M_0^k}^{M^k} \sum_{n=N_0^k}^{N^k} a_{mn}^k W_m^k(x) P_n^k(y) \quad , \quad k = I, II, III, \tag{3.1}$$

and

$$G^I(y) = G^{II}(y) = \tfrac{1}{2}y(3-y^2) \quad , \quad G^{III}(y) = G^I(2y);$$

$$M_0^I = N_0^I = M_0^{II} = N_0^{II} = M_0^{III} = 2, \; N_0^{III} = 4 \; .$$

The polynomials $\{W_m^k(x)\}$ and $\{P_n^k(y)\}$ are modified shifted Chebyshev polynomials which satisfy identically all the boundary conditions with the exception of the conditions along the vertical wall CD. For example, in region I we have

$$W_m^I(x) = T_m^I(x) + \alpha_m^I T_1^I(x) + \beta_m^I T_0^I(x) \quad , \quad 2 \le m \le M^I \; , \tag{3.2}$$

where $T_m^I(x)$, $0 \le m \le M^I$, are shifted Chebyshev polynomials on $[-h_1, 0]$ defined by

$$T_m^I(x) = T_m\left(\frac{2x + h_1}{h_1}\right) \; ,$$

and α_m^I and β_m^I are given by

$$\alpha^I{}_m = (-1)^m m^2 \; , \; \beta^I{}_m = (-1)^m(m^2 - 1) \; , \quad 2 \le m \le M^I \; .$$

Similarly, we can show that

$$P_n^I(y) = \tilde{T}^I{}_n(y) + \tilde{\alpha}_n^I \tilde{T}_1^I(y) + \tilde{\beta}^I{}_n \tilde{T}^I{}_0(y) \quad , \quad 2 \le n \le N^I \; , \tag{3.3}$$

where $\tilde{T}_n^I(y)$, $0 \le n \le N^I$, are the shifted polynomials on $[1/2, 1]$ defined by

$$\tilde{T}_n^I(y) = T_n(4y-3)$$

and $\tilde{\alpha}_n^I$, $\tilde{\beta}_n^I$ are given by

$$\tilde{\alpha}^I_n = -n^2 - 1, \quad \tilde{\beta}^I_n = n^2 - 1 \ .$$

The basis functions in regions II and III are defined in a similar fashion.

4. Collocation Strategy

The coefficients $\{a^k_{mn}\}$, k = I, II, III must be determined for the spectral approximations (3.1) to be defined everywhere in the truncated flow region. Suppose that at the beginning of a Newton step we have approximations a^{*k}_{mn} to the expansion coefficients of $\psi^k(x,y)$. The numerical solution of the linearized equation (2.9) determines the expansion coefficients $(\delta a)^k_{mn}$ of ϕ^k, the correction to ψ^k, in region k for k = I, II, III. We describe in some detail the process of calculating the coefficients $(\delta a)^k_{mn}$. Once these are known, the updated approximation to the stream function is just $\psi^{*k} + \phi^k$, which can be found by simply adding together a^{*k}_{mn} and $(\delta a)^k_{mn}$ for each value of m, n and k.

The linearized partial differential equation (2.9) is collocated in each subdomain and the approximations in the three subregions are patched by imposing the correct order of weak continuity across the subregion interfaces. The collocation points in a subdomain are chosen to be those points at which the Chebyshev polynomial of highest degree used in the representation in that subdomain attains its extreme values. This choice gives rise to optimal approximation properties of smooth functions. It can also be shown that when Gaussian quadrature rules based on these points are used to evaluate the integrals appearing in Galerkin formulations of certain differential equations, then the resulting equations are equivalent to those determined by collocating the differential equation at the same set of points. Thus, in region I, for example, the collocation points are given by

$$x^I_i = \frac{h_1(x_i - 1)}{2} \quad , \quad y^I_j = \frac{y_j + 3}{4} \quad , \tag{4.1}$$

where

$$x_i = -\cos\left(\frac{i\pi}{M^I}\right) \quad , \quad 0 \le i \le M^I \quad ,$$

$$y_j = -\cos\left(\frac{j\pi}{N^I}\right) \quad , \quad 0 \le j \le N^I \quad .$$

Boundary conditions

Due to the choice of modified Chebyshev polynomials as trial functions in the expansions (3.1) the boundary conditions are automatically satisfied except along $x = 0$ $(1/2 \le y \le 1)$ in region I. Along this portion of the boundary there are N^I+1 collocation points. We deduce from (3.1) that ψ and ψ_x are polynomials of degree N^I in y along $x = 0$ $(1/2 \le y \le 1)$ each depending on $N^I - 1$ degrees of freedom. Therefore collocation of the boundary conditions at the $N^I - 1$ distinct points $(0, y^I_j)$, $j = 0, 1, \ldots, N^I - 2$, ensures that the no-slip boundary conditions are satisfied identically along this part of the boundary.

Interface continuity conditions

We impose C^3 continuity of the stream function across the interfaces between the subdomains. These conditions are collocated in such a way so as to yield approximations that are pointwise C^1 continuous across the interfaces, but whose second and third normal derivatives are continuous only at the interface collocation points. If the initial approximation to the solution of the problem satisfies these conditions then these continuity conditions are imposed on ϕ at each Newton step.

Let us examine the interface $y = 1/2(-h_1 \le x \le 0)$ between subregions I and II, for example. Along $y = 1/2$, ϕ^I and ϕ^{II} are polynomials of degree M^I each possessing $M^I - 1$ degrees of freedom. We collocate at a sufficient number of points on the interface to ensure that $\phi^I - \phi^{II}$ is identically zero. This is achieved if we impose

$$\phi^I(x^I_i, \tfrac{1}{2}) - \phi^{II}(x^I_i, \tfrac{1}{2}) = 0 \quad , \quad i = 2, 3, \ldots, M^I - 2, \tag{4.2}$$

and, in addition,

$$\phi^{II}(0, \tfrac{1}{2}) = 0, \quad \frac{\partial\phi^{II}}{\partial x}(0, \tfrac{1}{2}) = 0 \quad . \tag{4.3}$$

Similarly, continuity of ϕ_y is obtained by imposing

$$\frac{\partial\phi}{\partial y}(x_i^I, \tfrac{1}{2}) - \frac{\partial\phi^{II}}{\partial y}(x_i^I, \tfrac{1}{2}) = 0 , \quad i = 2, 3, \ldots, M^I - 2, \tag{4.4}$$

and

$$\frac{\partial\phi^{II}}{\partial y}(0, \tfrac{1}{2}) = 0 , \quad \frac{\partial^I\phi^{II}}{\partial x\partial y} = 0 \quad . \tag{4.5}$$

The conditions which represent the continuity of the second and third normal derivatives of ϕ are collocated at the $M^I - 3$ points $(x_i^I , 1/2)$, $i = 2, 3, \ldots, M^I - 2$. Thus, these derivatives are not pointwise continuous across the interface. Moffatt (1964) shows that the leading term in the singular expansion of the Stokes solution about the re-entrant corner (0, 1/2) is $0(r^\lambda)$ where $\lambda \simeq 1.5445$ and so it would be inconsistent to impose pointwise continuity of these higher derivatives across the interface.

The same collocation strategy is applied across the interface $x = 0$ $(0 \leq y \leq 1/2)$ between subregions II and III. As a result the correction ϕ and its *normal derivative are pointwise continuous* across the interfaces. The updated approximations to the stream function also possess the same order of pointwise continuity.

Linearized differential equation

The linearized differential equation (2.9) is collocated in each subregion at all points on the collocation grid with the exception of those on or one in from each subregion boundary, i.e.,

$$(x_i^k , y_j^k) , \quad i = 2, \ldots, M^k - 2, \; j = 2, \ldots, N^k - 2,$$

for k = I, II or III.

When the spectral collocation equations resulting from the boundary conditions, interface continuity conditions and differential equation are added together, they yield a total of $[(M^I - 1)(N^I - 1) + (M^{II} - 1)(N^{II} - 1) +$

$(M^{III}-1)(N^{III}-3)]$ linear equations which is equal to the number of unknown expansion coefficients $(\delta a)^k_{mn}$ for $k = I, II, III$. Therefore, provided the coefficient matrix is non-singular, this system of equations possesses a unique solution.

5. Direct Method of solution

The global system for the expansion coefficients is

$$G\underline{z} = \underline{r} \tag{5.1}$$

where

$$G = \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline \multicolumn{2}{c|}{B} & 0 \\ \hline 0 & C & 0 \\ \hline 0 & \multicolumn{2}{c}{D} \\ \hline 0 & 0 & E \end{array}\right], \quad \underline{z} = \left[\begin{array}{c} (\delta\underline{a})^{I} \\ \hline (\delta\underline{a})^{II} \\ \hline (\delta\underline{a})^{III} \end{array}\right], \quad \underline{r} = \left[\begin{array}{c} \underline{r}_1 \\ \hline \underline{r}_2 \\ \hline \underline{r}_3 \\ \hline \underline{r}_4 \\ \hline \underline{r}_5 \end{array}\right] \tag{5.2}$$

The vectors $(\delta\underline{a})^{I}$, $(\delta\underline{a})^{II}$ and $(\delta\underline{a})^{III}$ contain the unknown coefficients in the expansions in subregions I, II and III, respectively. The first, third and fifth blocks of rows in (5.2) correspond to the collocation of the linearized differential equation and boundary conditions not satisfied by the approximations in subregions I, II, and III, respectively. The second and fourth blocks of rows correspond to the

collocation of the interface continuity conditions between subregions I and II, and II and III, respectively.

The system (5.1) is solved using an application of a standard production code designed for the solution of almost block diagonal systems (Brankin and Gladwell (1990)). In a recent survey of direct methods for solving systems of equations arising from spectral domain decomposition methods, Karageorghis and Phillips (1990) found this solver to be superior to other techniques with regard to cost, stability and storage. The code uses a modified column elimination procedure with alternate row and column pivoting based on an algorithm originally described in Varah (1976) and Diaz, Fairweather and Keast (1983) and is intended to solve systems of the form shown in Figure 2. These systems comprise rectangular blocks along the diagonal and are such that no three successive blocks have columns in common.

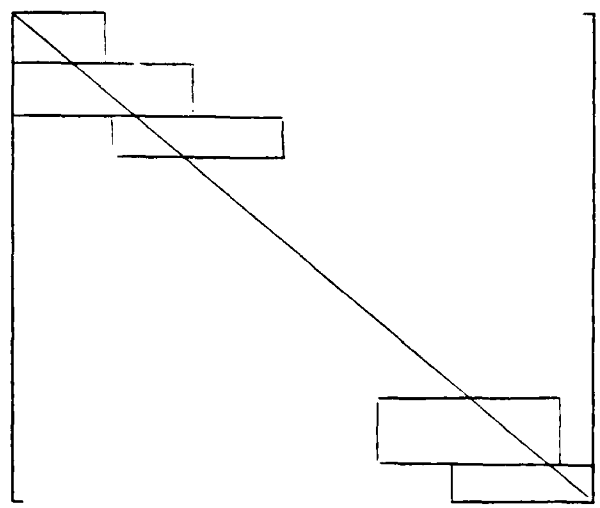

Fig.2 Almost block diagonal form.

The global spectral collocation matrix G in (5.1) may be written in almost block diagonal form in the obvious way:

$$G = \left[\begin{array}{cc|c} P_1 & 0 & \\ \hdashline P_2 & Q_2 & 0 \\ \hline Q_1 & R_1 & \\ \hdashline & & 0 \\ & R_2 & \\ 0 & & \\ \hdashline & R_3 & S_2 \\ \hline & S_1 & T_1 \\ \hdashline 0 & & \\ & 0 & T_2 \end{array}\right] \qquad (5.3)$$

The matrix G has five non-zero blocks, namely,

$$[P_1] \; , \begin{bmatrix} P_2 & Q_2 \\ Q_1 & R_1 \end{bmatrix} \; , \; [R_2] \; , \begin{bmatrix} R_3 & S_2 \\ S_1 & T_1 \end{bmatrix} \quad \text{and} \quad [T_2] \; .$$

Unfortunately the form (5.3) is not of the structure required by the almost block diagonal solver due to too much overlap between blocks 2 and 3, and 3 and 4. However, the transpose of G is of the required form, with three blocks, namely,

$$\left[P^T \mid Q_1^T\right] \; , \; \left[Q_2^T \mid R^T \mid S_1^T\right] \quad \text{and} \quad \left[S_2^T \mid T^T\right] \; .$$

One may, therefore, decompose the transpose of the global matrix G^T using the existing NAG subroutine F01LHF and subsequently solve using the transpose of the decomposed form of G^T, say $\tilde{G}^T$, the system

$$\left[\tilde{G}^T\right]^T \underline{z} = \underline{r} \qquad (5.4)$$

with the NAG subroutine F04LHF. Further details of the implementation may be found in Karageorghis and Phillips (1990).

6. Numerical Results

The results of numerical calculations of laminar flow through a 2:1 constricted channel are presented and comparisons made with the work of Dennis and Smith (1980) and Hunt (1989). Dennis and Smith (1980) use a finite difference discretization of the stream function-vorticity formulation of the Navier-Stokes equations on a uniform grid. An upwind differencing scheme of Dennis and Hudson (1978) is used to approximate the vorticity transport equation. This is essential for reasonably high values of Re in order to maintain the diagonal dominance in the approximating sets of difference equations. The loss of diagonal dominance presents difficulties in the convergence of iterative solution techniques. If upwinding is not used then diagonal dominance can only be established either by sufficiently decreasing the mesh size, which may be prohibitively expensive, or by using a non-uniform grid. Hunt (1989) also uses a finite difference discretization of the stream function-vorticity formulation but on a non-uniform grid. The vorticity unknowns are eliminated to give a system solely in terms of unknown values of the stream function. Hunt (1989) investigates the application of artificial viscosity or upwinding but contrary to expectations finds that the scheme with upwinding fails to converge for $Re > 500$ even though one would expect the addition of a 'viscous' like term to have a stabilizing effect. Further for a given value of Re convergence becomes increasingly more difficult on successively finer grids. The opposing conclusions on the application of upwind differencing must be due to the different nature of the grids used in these studies. The ability of these two methods and the spectral algorithm to describe the main features of the flow is examined.

The solution of the Stokes problem ($Re = 0$) is used as the initial approximation to the solution of the Navier-Stokes problem for $0 < Re \leq 150$. However, for $Re \leq 150$ the Newton process is robust enough to converge from an initial approximation that is zero everywhere. For $Re > 150$ continuation in Re is required for convergence in increments of Re of 50. This means, for example, that the converged numerical solution for $Re = 150$ is used as the initial approximation when calculating the numerical solution for $Re = 200$.

The Newton process is terminated when the maximum residual is less than 10^{-7} in magnitude. This invariably occurs after six Newton iterations. The expansion coefficients are also checked for convergence. The last two iterations

are found not to affect the first eight significant digits of all the expansion coefficients.

Numerical experiments are performed on a number of collocation grids. This is necessary in order to ensure that the approximations obtained are not strongly grid dependent. The following discretization parameters define the collocation grids used in this work and the total number of degrees of freedom:

(1) $M^I = M^{II} = 24$, $N^I = 20$, $N^{II} = N^{III} = 16$, $M^{III} = 16$ (977 degrees of freedom);

(2) $M^I = M^{II} = 24$, $N^I = N^{II} = N^{III} = 20$, $M^{III} = 24$ (1265 degrees of freedom);

(3) $M^I = M^{II} = 24$, $N^I = N^{II} = N^{III} = 20$, $M^{III} = 32$ (1401 degrees of freedom);

(4) $M^I = M^{II} = 24$, $N^I = N^{II} = N^{III} = 20$, $M^{III} = 40$ (1537 degrees of freedom).

For Re $\leq$ 100 good agreement is obtained on all four grids. For $100 \leq Re \leq 300$ good agreement is reached on grids (2) — (4). Thereafter, for Re $\geq$ 300, grids (3) and (4) produced almost identical results.

The upstream and downstream truncation lengths are chosen to be 1.0 and 3.5, respectively. It is necessary to experiment with the number of degrees of freedom in subregion III since for an insufficient number the flow is not adequately resolved particularly near the downstream recirculation region. Dennis and Smith (1980) admit that they do not resolve this feature of the flow and that the use of very fine grids is necessary to recover the true behaviour. However, from their numerical calculations they are able to imply that the fluid separates downstream of the constriction.

The behaviour of the vorticity along the downstream channel wall ($y = \frac{1}{2}$, $x \geq 0$) indicates whether the domain has been truncated far enough downstream. As $x \to \infty$ the vorticity $\zeta = -\nabla^2\psi \to 12$. If $\zeta(x, \frac{1}{2})$ settles down to this value then the truncation length is adequate. The value of $\zeta(x, \frac{1}{2})$ is plotted in Figs. 3(a) — (c) for Re = 0, 100, 500, respectively. These figures suggest that the exit length is

suitable for the range of Reynolds numbers considered here. Dennis and Smith (1980) use downstream truncation lengths of 2 and 2.5 and obtain excellent agreement in their results which demonstrates that the numerical solutions obtained with a truncation length of 2 are satisfactory. Hunt (1989) uses a transformation of the independent variables to set the downstream boundary at $x \sim 1000$. Instead of experimenting with truncation lengths he needs to determine appropriate mapping parameters which appear in the transformation.

Contours of the stream function are presented in Fig. 4(a) — (i) for Re = 0, 10, 50, 100, 150, 200, 300, 400 and 500, respectively, in the region $-1 \leq x \leq 1$, $0 \leq y \leq 1$. The vortex in the salient corner diminishes in size from Re=0 until around Re = 45 and then increases slowly as Re is increased further. Let L_V denote the distance between the point where the separation line meets the top of the channel and the salient corner. The value of L_V is recorded in Table I for different methods and for a range of values of Re. The results of Dennis and Smith (1980) are obtained by means of two successive h^2 — extrapolation operations on information calculated on grids of mesh lengths 1/10, 1/20 and 1/40. Hunt (1989) uses a transformed grid with 48 x 128 points. The values of L_V in columns (b) and (c) agree to within 5%. For the range of values of Re considered here the scheme of Hunt (1989) that uses artificial viscosity gives results that are closer to those in column (b) than the scheme without artificial viscosity.

Close-up views, $-\frac{1}{8} \leq x \leq 0$, $\frac{7}{8} \leq y \leq 1$, of the salient corner are given in Fig. 5(a) — (e) for Re = 10, 50, 100, 150 and 200, respectively. The first closed streamline corresponds to a contour height of $1 + 10^{-5}$ and the remaining ones differ by multiples of 10^{-5}. The interesting feature in these plots is the second vortex appearing close to the corner. In the Stokes case Moffatt (1964) predicts an infinite sequence of eddies running into the corner. The size of this second vortex grows moderately as Re is increased. Dennis and Smith (1980) only just detect the second vortex at values of Re of at least a thousand and then only on extremely fine meshes of size 1/80.

A small recirculation region downstream of the constriction is observed in Fig. 4 (d) for Re = 100. This region suddenly grows when Re is increased to 200. In fact when the stream function is calculated at intermediate values of Re this downstream recirculation region grows suddenly at a value of Re around 175. A

magnification of this region is shown in Fig. 6 (a) - (d) for Re = 200, 300, 400 and 500, respectively. In these figures the stream function is contoured in the rectangle $0 \leq x \leq 1$, $\frac{2}{5} \leq y \leq \frac{1}{2}$. These figures give a good description of the way in which the vortex develops as Re is increased. In Fig. 6 the fluid is seen to separate at the corner and not slightly to the right as Hunt (1989) observes. Further the length of the downstream recirculation regions are significantly longer than those calculated by Hunt (1989) as can be seen in Table II. Dennis and Smith (1980) do not detect this region although they say that its existence is implied as the grid is refined. For Re = 500 they predict that separation occurs at a point in the interval $0 < x < 0.3$ with reattachment at a point beyond $x = 1.2$. For this value of Re their prediction of the reattachment point is closer to our calculated value than the value obtained by Hunt (1989).

7. Concluding Remarks

The steady planar two-dimensional laminar flow of an incompressible fluid through an abruptly contracting channel is considered for moderate values of the Reynolds number. The governing equation for the stream function is linearized using Newton's method and solved using spectral domain decomposition techniques. A conforming domain decomposition is used which in conjunction with a carefully constructed collocation strategy ensures that the resulting spectral approximations are globally C^1 - continuous. In addition, the spectral approximations are C^∞ except along subdomain interfaces. An efficient direct method for solving the spectral collocation equations at each stage of the Newton process is described.

At most six Newton iterations are required for convergence. For $0 < \mathrm{Re} \leq 150$ a converged numerical solution is obtained from an initial approximation that is zero everywhere. Continuation in Re in increments of 50 is used for $150 < \mathrm{Re} \leq 500$. The vortex in the salient corner decreases in size from Re = 0 to around Re = 45 and then grows slowly as Re is increased further. A small recirculation region just downstream of the constriction appears at Re = 100. This region suddenly grows as Re is increased to a value around 175. This recirculation region extends further downstream as Re is increased.

There is qualitative and quantitative agreement with Dennis and Smith (1980) in the bulk of the flow and the description of the salient corner vortex. Their

numerical solutions do not predict a downstream recirculation region although they say there is a hint of its existence as the grid is refined. Our calculations predict that the fluid separates at the reentrant corner not just to the right of it as does Hunt (1989) and that reattachment occurs further downstream than in his simulations.

This paper demonstrates that spectral domain decomposition techniques are capable of solving complex flow situations and resolving the main features of the flow. This is accomplished in an efficient manner using far fewer degrees of freedom than other methods of discretization.

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TABLE I

Re	(a)	(b)	(c)	(d)	(e)
0	0.285	0.290	0.284	-	-
10	0.150	0.148	0.155	-	-
25	0.129	0.125	-	-	-
50	0.129	0.123	0.129	-	-
75	0.135	0.135	-	-	-
100	0.143	0.140	0.144	-	-
125	0.154	0.150	-	0.164	0.168
150	0.160	0.155	-	-	-
200	0.183	0.183	-	-	-
250	0.210	0.205	-	0.209	0.227
300	0.223	0.223	-	-	-
350	0.235	0.233	-	-	-
400	0.244	0.244	-	-	-
450	0.251	0.255	-	-	-
500	0.260	0.265	0.266	0.260	0.308

The length of the upstream vortex as a function of Re for (a) spectral collocation method on grid (3), (b) spectral collocation method on grid (4), (c) extrapolated finite difference scheme of Dennis and Smith (1980), (d) finite difference scheme of Hunt (1989) with artificial viscosity, (e) finite difference scheme of Hunt (1989) without artificial viscosity.

TABLE II

Re	(a)	(b)	(c)
150	0.088	-	-
200	0.363	-	-
250	0.500	0.106	0.096
300	0.538	-	-
350	0.700	-	-
400	0.738	-	-
450	0.924	-	-
500	0.995	0.406	0.406

The length of the downstream vortex as a function of Re for (a) spectral collocation method on grid (4), (b) finite difference scheme of Hunt (1989) with artificial viscosity, (c) finite difference scheme of Hunt (1989) without artificial viscosity.

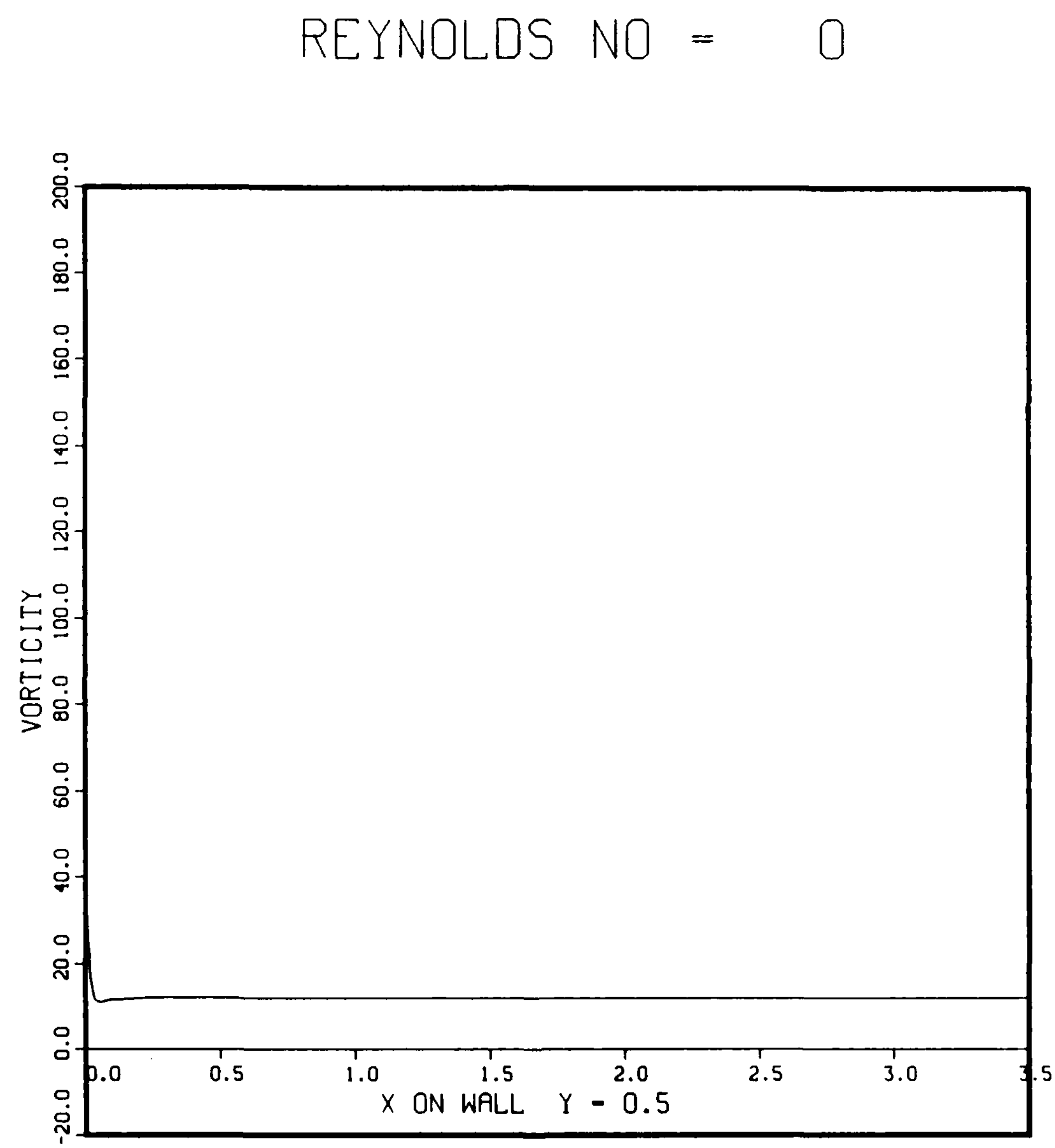
CAPTIONS FOR FIGURES

Fig. 3. Wall vortcity $\zeta(x,1/2)$ for $x > 0$, for various Reynolds numbers Re.

Fig. 4. Streamfunction contours in $-1 \leq x \leq 1$, $0 \leq y \leq 1$, for various Reynolds numbers Re.

Fig. 5. Close-up views of the salient corner streamlines on $-1/8 \leq x \leq 0$, $7/8 \leq y \leq 1$ for various Reynolds numbers Re.

Fig. 6. Close-up views of the streamlines in the region $0 \leq x \leq 1$, $\frac{2}{5} \leq y \leq \frac{1}{2}$ for various Reynolds numbers Re.


REYNOLDS NO = 0
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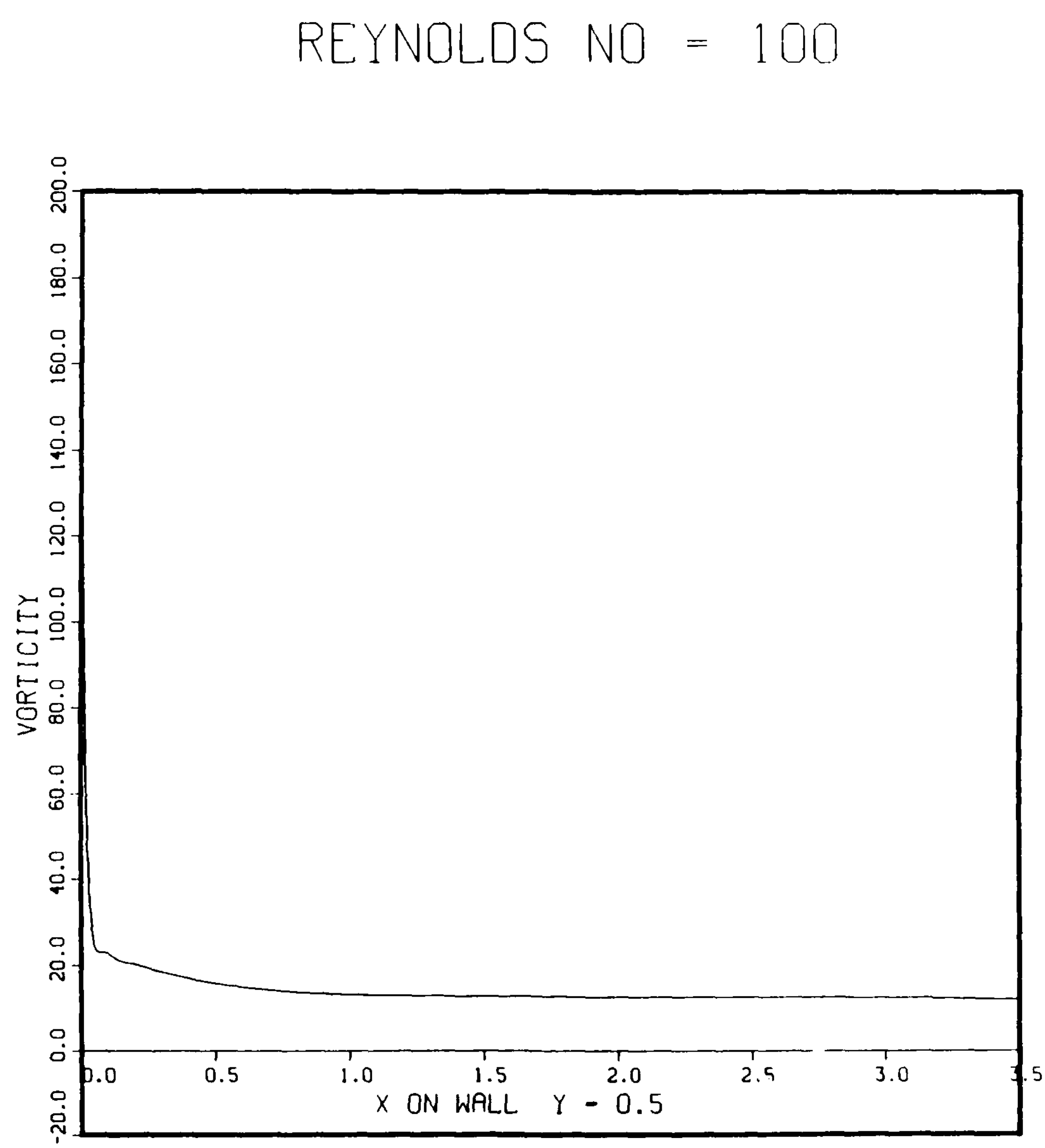
200.0
180.0
160.0
140.0
120.0
100.0
80.0
60.0
40.0
20.0
0.0
-20.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
X ON WALL Y - 0.5

Fig. 3(a)


REYNOLDS NO = 100
VORTICITY
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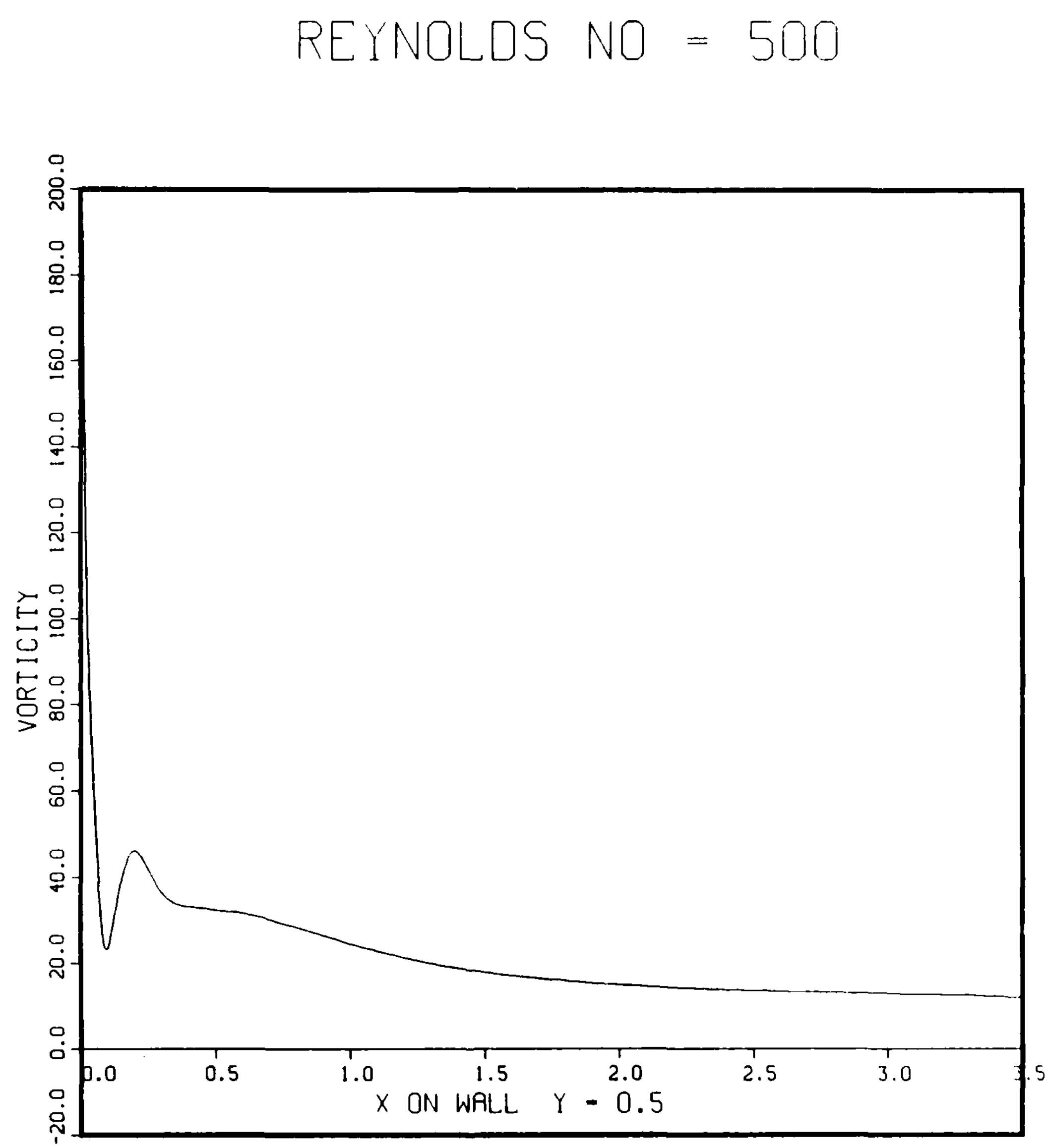
160.0
140.0
120.0
100.0
80.0
60.0
40.0
20.0
0.0
-20.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
X ON WALL Y - 0.5

Fig. 3(b)


REYNOLDS NO = 500
VORTICITY
200.0
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140.0
120.0
100.0
80.0
60.0
40.0
20.0
0.0
-20.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
X ON WALL Y - 0.5

Fig. 3(c)

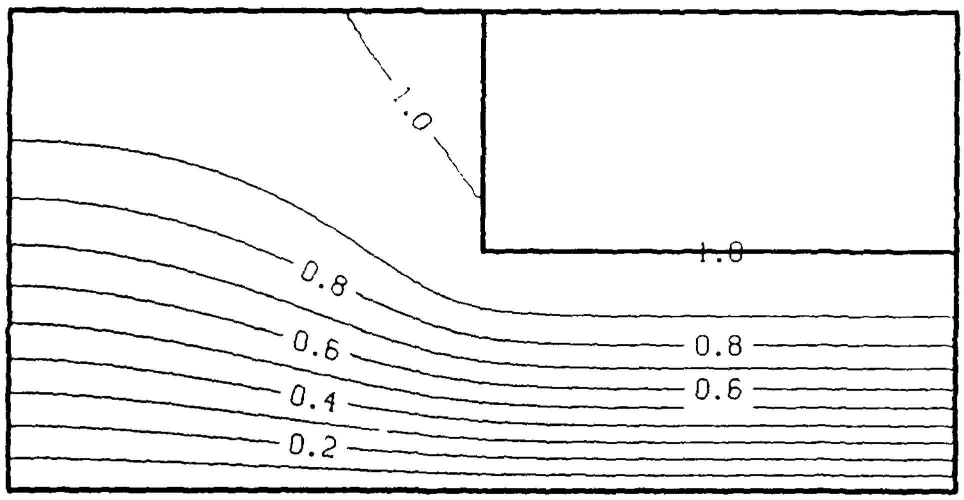


Fig. 4(a) Re = 0

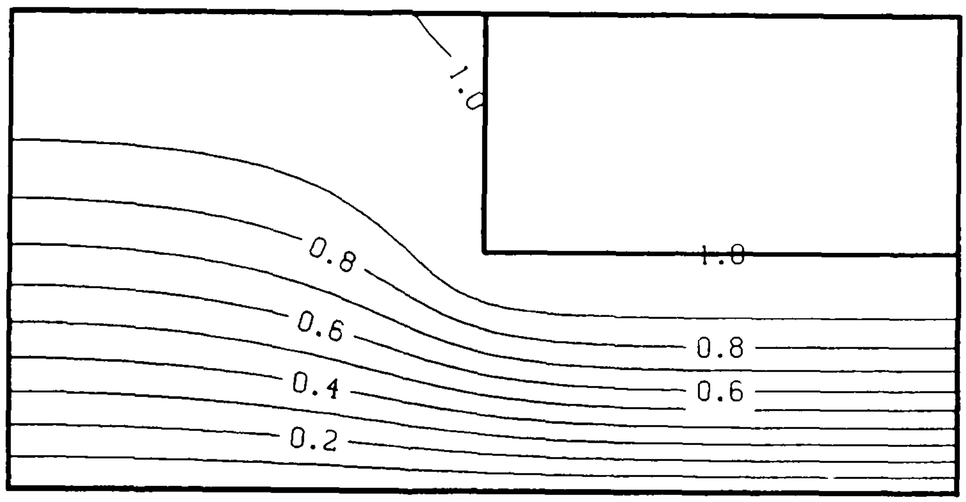


Fig. 4(b) Re = 10

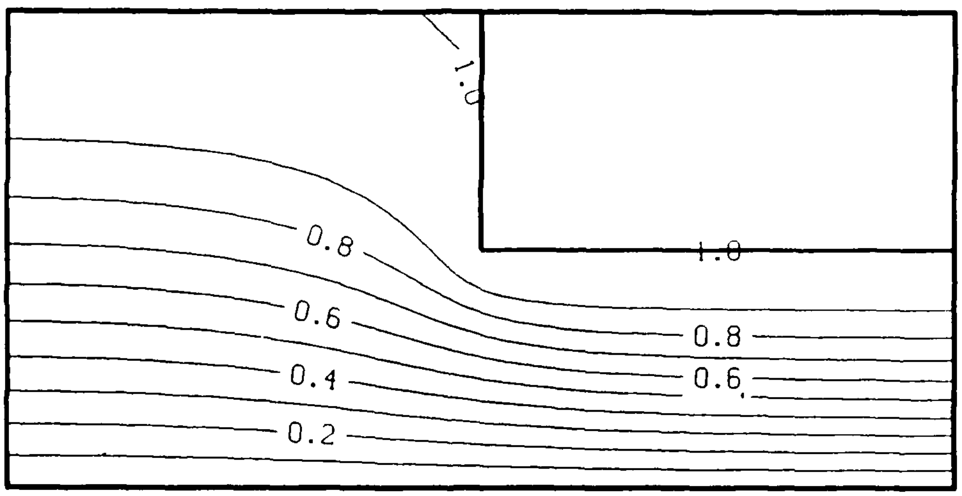


Fig. 4(c)   Re = 50

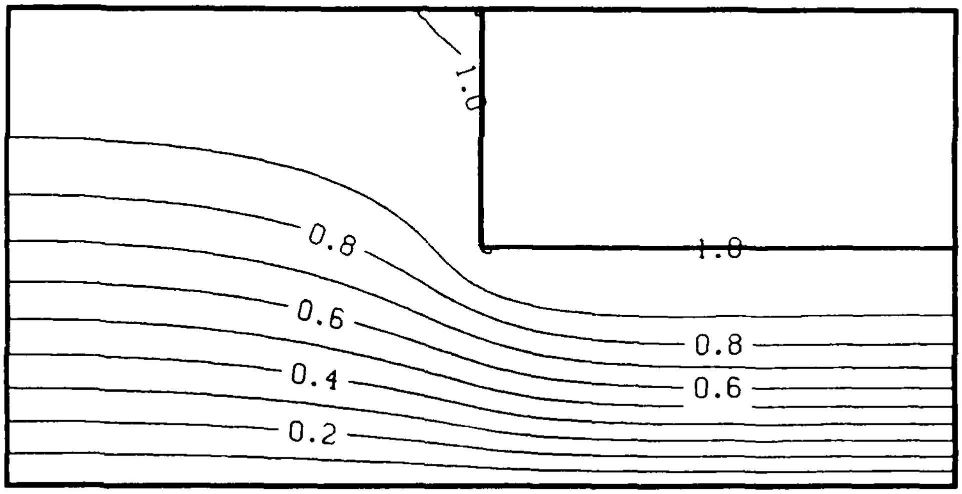


Fig. 4(d) Re = 100

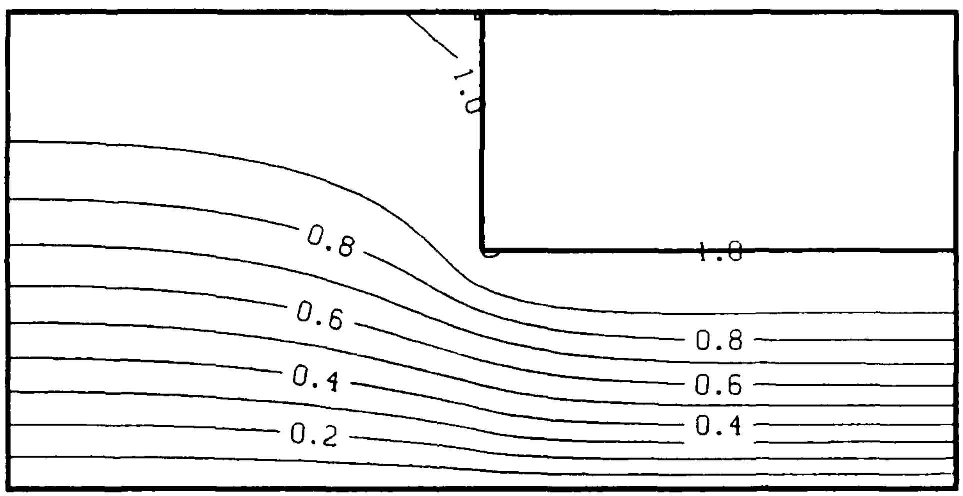
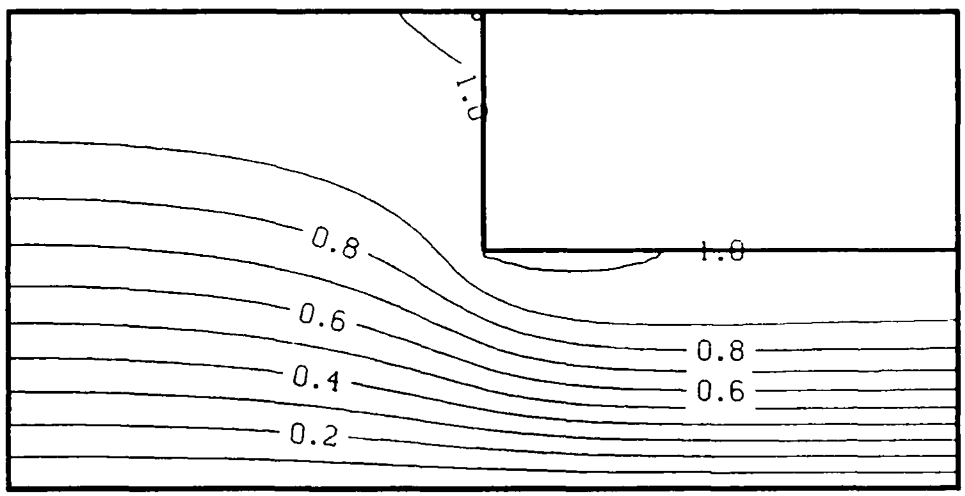


Fig. 4(e) Re = 150



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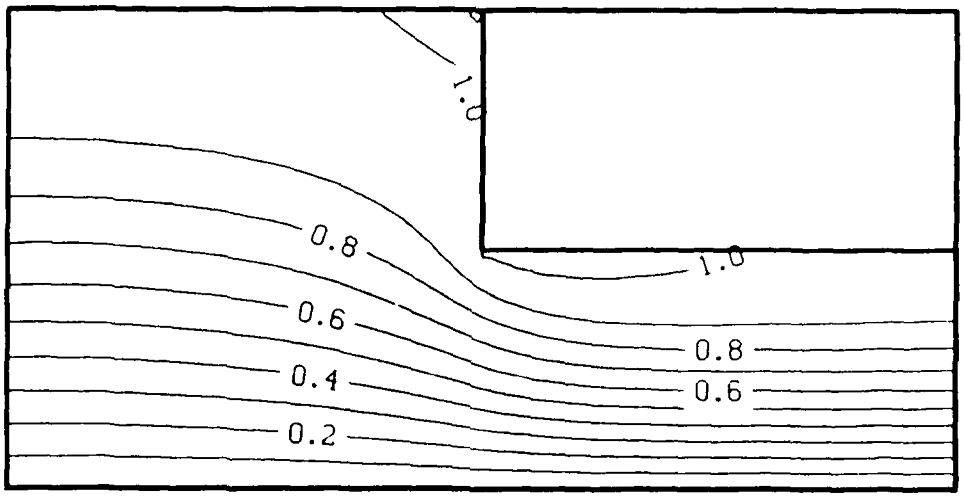

Fig. 4(g) Re = 300

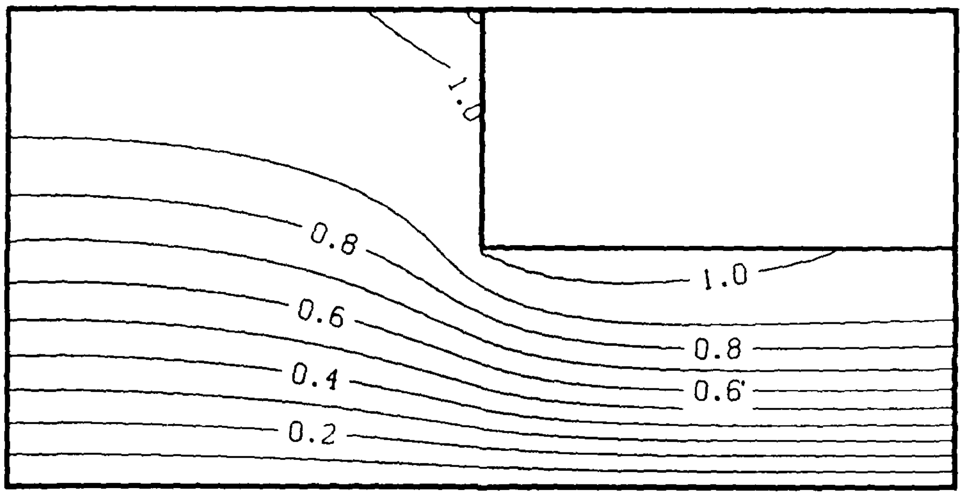

Fig. 4(h) Re = 400

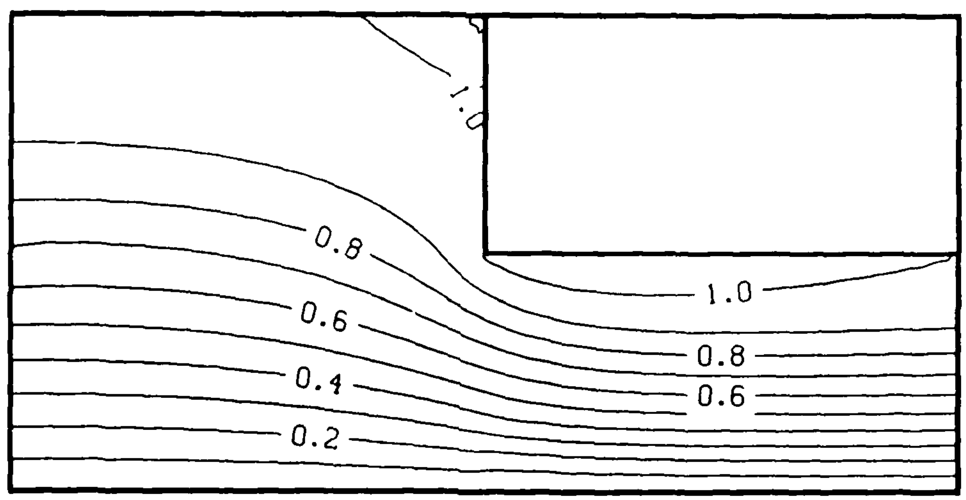

Fig. 4(i)   Re = 500

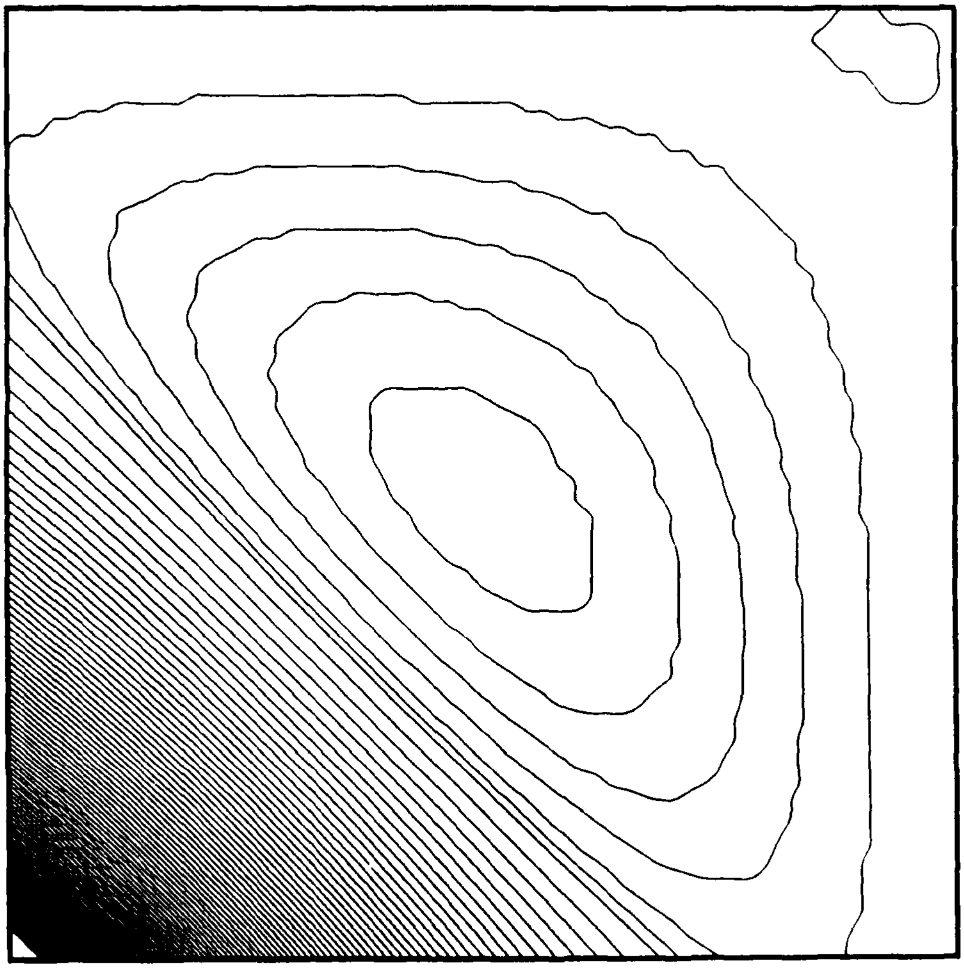

Fig. 5(a) Re = 10

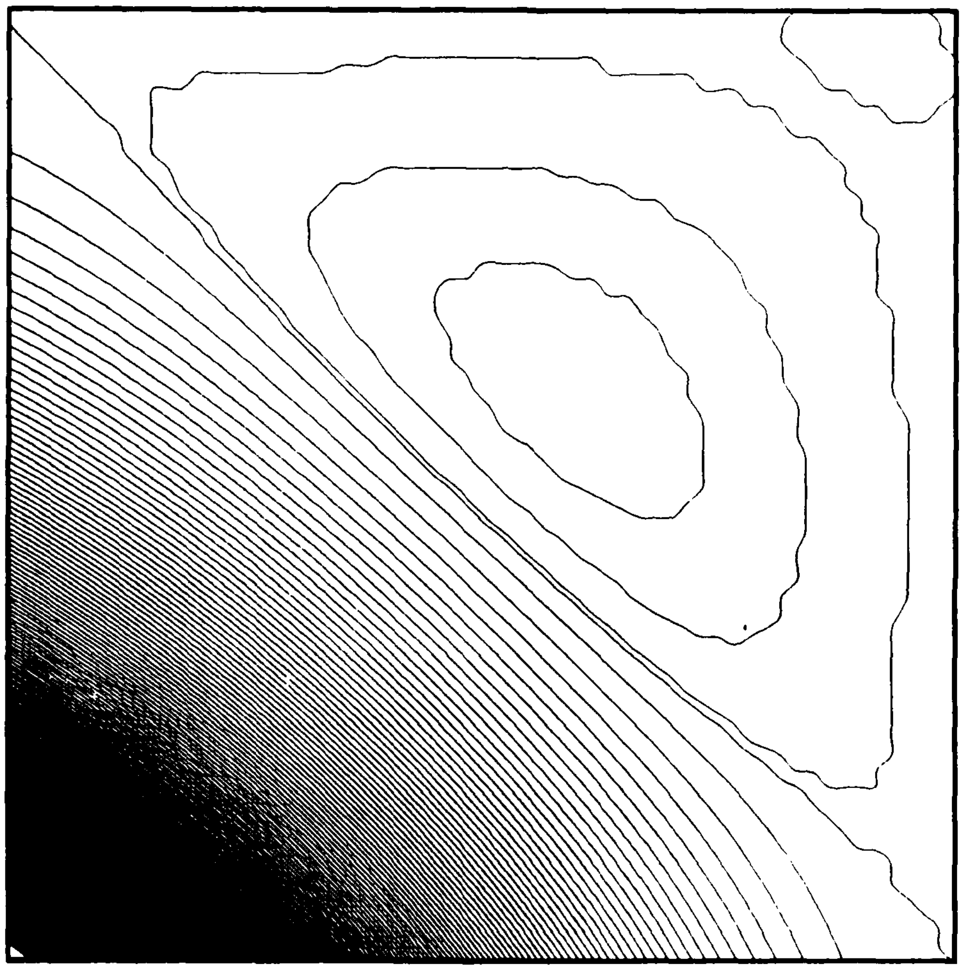

Fig. 5(b)   Re = 50

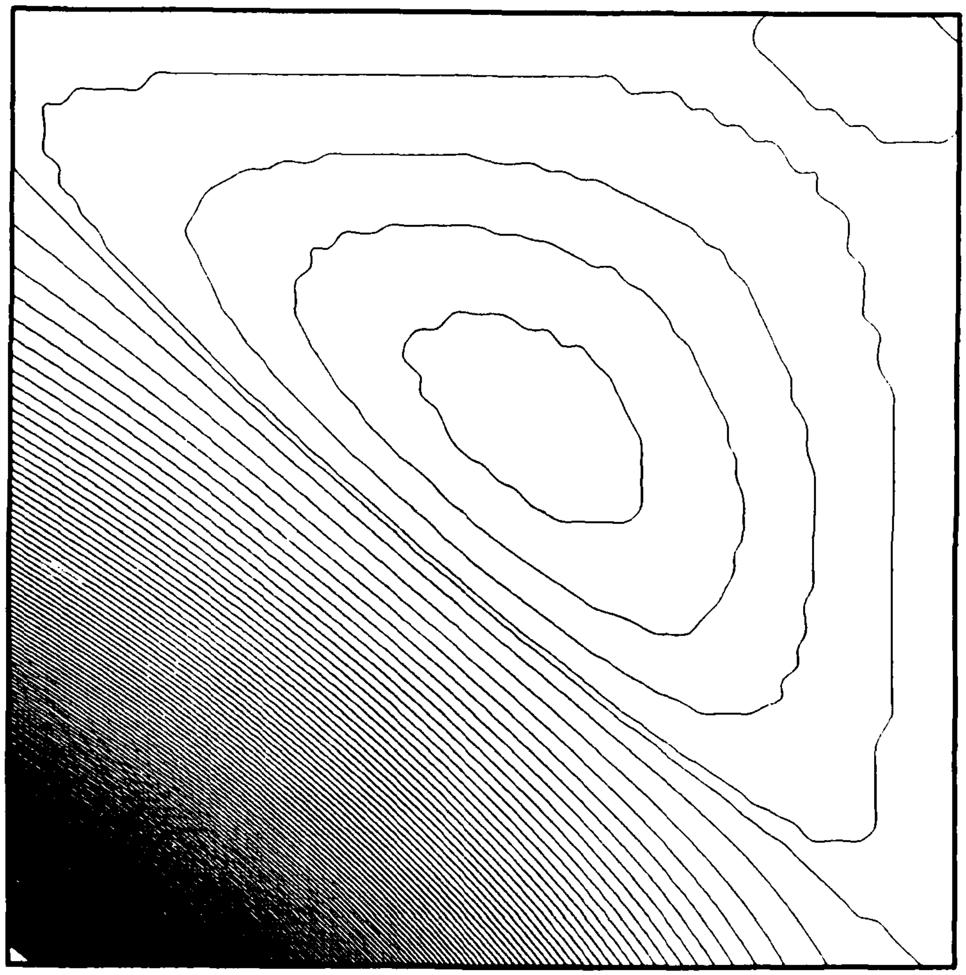

Fig. 5(c) Re = 100

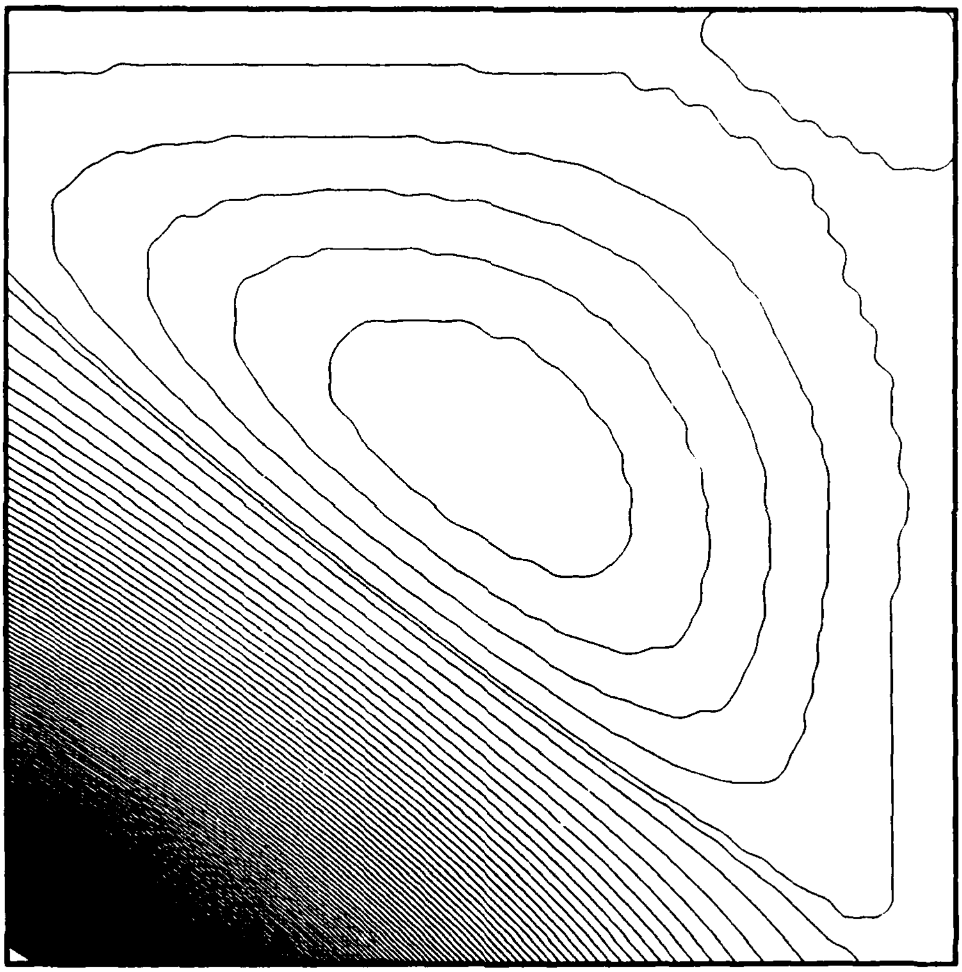

Fig. 5(d) Re = 150

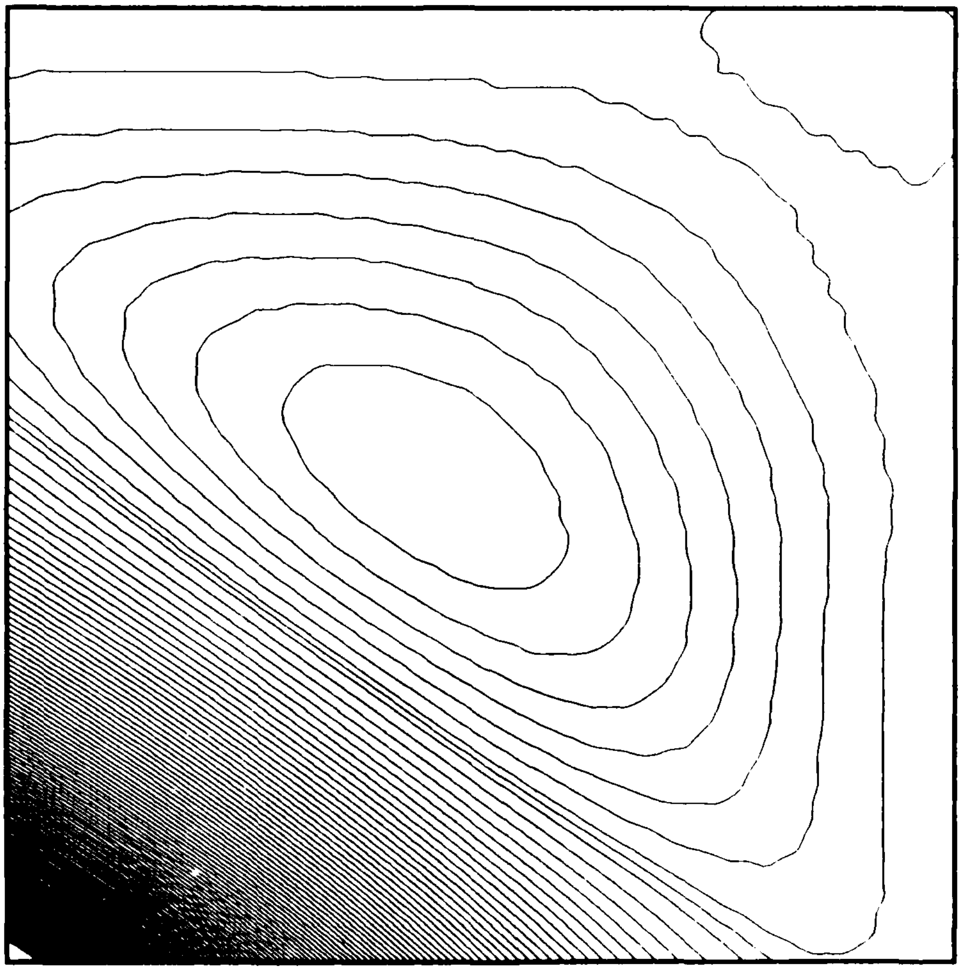

Fig. 5(e) Re = 200

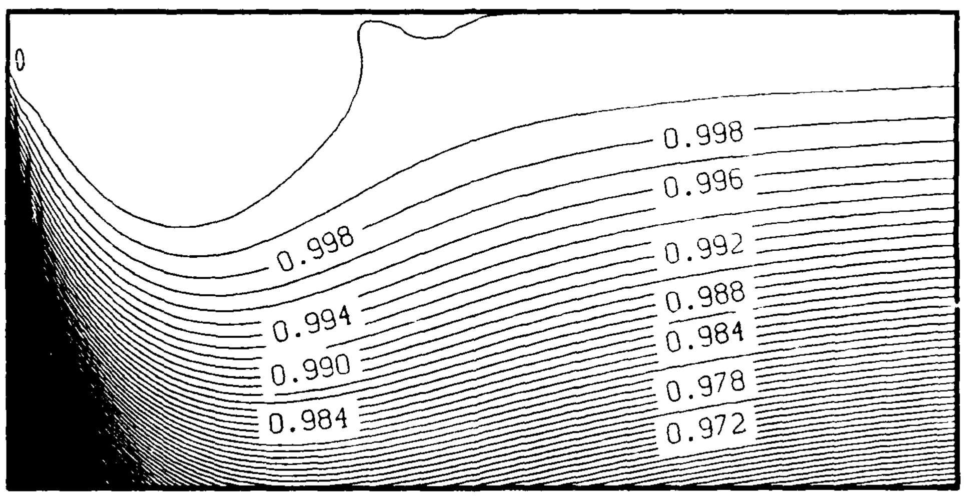

Fig. 6(a)   Re = 200

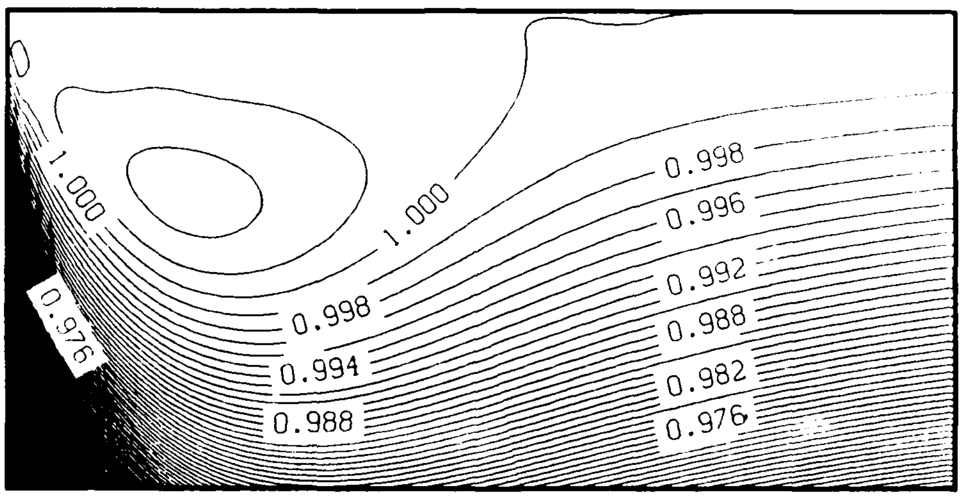

Fig. 6(b)   Re = 300

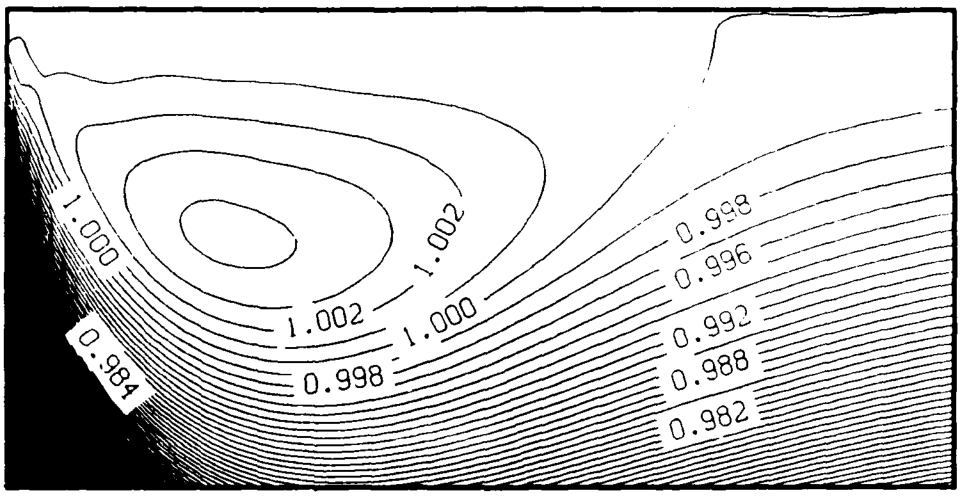

Fig. 6(c) Re = 400

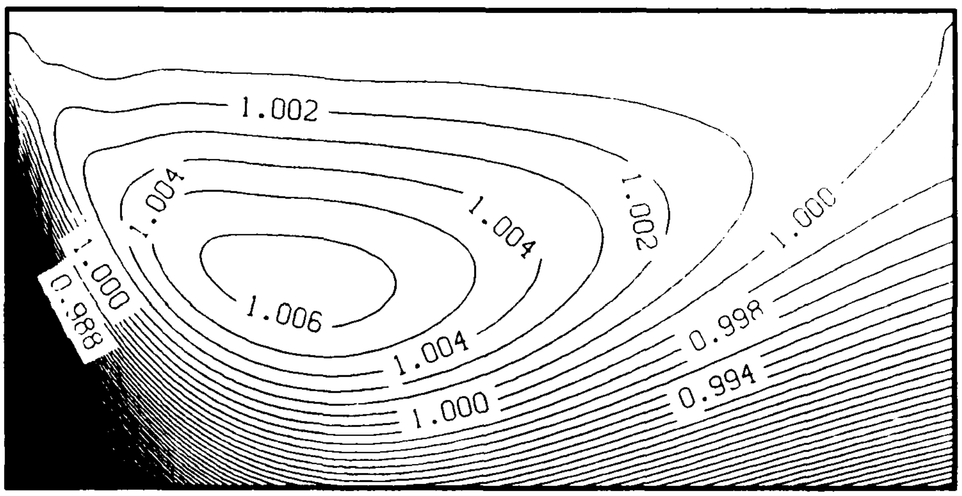

Fig. 6(d)   Re = 500

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Final Report

16. Abstract

The numerical simulation of steady planar two-dimensional, laminar flow of an incompressible fluid through an abruptly contracting channel using spectral domain decomposition methods is described. The key features of the method are the decomposition of the flow region into a number of rectangular subregions and spectral approximations which are pointwise C^1 continuous across subregion interfaces. Spectral approximations to the solution are obtained for Reynolds numbers in the range (0, 500). The size of the salient corner vortex decreases as the Reynolds number increases from 0 to around 45. As the Reynolds number is increased further the vortex grows slowly. A vortex is detected downstream of the contraction at a Reynolds number of around 175 that continues to grow as the Reynolds number is increased further.

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